回転円筒座標系における電磁流体力学の方程式

本稿では，回転磁場シミュレーションに必要となる回転円筒座標系$\left(\hat{r},\hat{\theta},\hat{z}\right)$における電磁流体力学の基礎方程式をまとめておく．特に，静止座標系と回転座標系における違いに焦点を当てる．図１に示すように，単位ベクトルは，半径方向に$\vec{e}_r$，回転方向に$\vec{e}_\theta$，軸方向に$\vec{e}_z$とする．角速度ベクトルは，角速度Ωを用いて，次式で表される．

$$\vec{\Omega}=\Omega\vec{e}_z \tag{1}$$

静止座標系と回転座標系を区別するために，ここでは，$\hat{\cdot}$は回転座標系からみた変数に付与するものと約束する．ただし，表１に示すように，空間微分は座標系によらず不変であることから，$\left(\hat{r},\hat{\theta},\hat{z}\right)$には，$\hat{\cdot}$は以後は特に断らない限りは付与しないとする．

表１　変数の座標系依存性

座標系で変わるもの	座標系で変わらないもの
速度（周方向成分），時間微分，電位，電場，運動方程式	磁場，ベクトルポテンシャル，空間微分，電流密度（オームの法則），誘導方程式，エネルギー方程式

以下，式(2)から(5)は，回転座標系と慣性座標系の関係をまとめる．

＜回転角＞

$$\hat{\theta}=\theta-\Omega t \tag{2}$$

＜速度ベクトル＞

$$\hat{\vec{u}}=\vec{u}-\vec{\Omega}\times\vec{r}=\vec{u}-r\Omega\vec{e}_z\times\vec{e}_r=\vec{u}-r\Omega\vec{e}_\theta \quad\rightarrow\quad \begin{cases}\hat{u}_r=u_r\\ \hat{u}_\theta=u_\theta-r\Omega\\ \hat{u}_z=u_z\end{cases} \tag{3}$$

＜時間の偏導関数＞　※付録を参照

$$\frac{\partial}{\partial\hat{t}}=\left[\frac{\partial}{\partial t}+\left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}-\vec{\Omega}\times\right]=\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\theta} \tag{4}$$

＜電位＞

$$\hat{\varphi}_e=\varphi_e-a_\theta\Omega r \tag{5}$$

式(5)におけるa_θは，磁場のベクトルポテンシャルの周方向成分である．

静止座標系

＜質量保存式＞

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}=0 \tag{6}$$

＜運動方程式＞

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{{u_\theta}^2}{r}\right)$$
$$=-\frac{\partial p}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right) \tag{7}$$
$$+\frac{1}{\mu_m}\left(\frac{b_\theta}{r}\frac{\partial b_r}{\partial \theta}+b_z\frac{\partial b_r}{\partial z}-\frac{b_\theta b_\theta}{r}-b_\theta\frac{\partial b_\theta}{\partial r}-b_z\frac{\partial b_z}{\partial r}\right)+\rho g_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_\theta u_r}{r}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right) \tag{8}$$
$$+\frac{1}{\mu_m}\left(b_r\frac{\partial b_\theta}{\partial r}+b_z\frac{\partial b_\theta}{\partial z}+\frac{b_\theta b_r}{r}-\frac{b_r}{r}\frac{\partial b_r}{\partial \theta}-\frac{b_z}{r}\frac{\partial b_z}{\partial \theta}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)$$
$$+\frac{1}{\mu_m}\left(b_r\frac{\partial b_z}{\partial r}+\frac{b_\theta}{r}\frac{\partial b_z}{\partial \theta}-b_r\frac{\partial b_r}{\partial z}-b_\theta\frac{\partial b_\theta}{\partial z}\right)+\rho g_z \tag{9}$$

＜エネルギー方程式＞

$$\frac{\partial T}{\partial t}+u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial \theta}+u_z\frac{\partial T}{\partial z}=\alpha\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right\} \tag{10}$$

＜電荷保存則＞

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rj_r\right)+\frac{1}{r}\frac{\partial j_\theta}{\partial \theta}+\frac{\partial j_z}{\partial z}=0 \tag{11}$$

＜Ohm の法則＞

$$j_r=\sigma\left(-\frac{\partial \varphi_e}{\partial r}-\frac{\partial a_r}{\partial t}+u_\theta b_z-u_z b_\theta\right) \tag{12}$$

$$j_\theta=\sigma\left(-\frac{1}{r}\frac{\partial \varphi_e}{\partial \theta}-\frac{\partial a_\theta}{\partial t}+u_z b_r-u_r b_z\right) \tag{13}$$

$$j_z=\sigma\left(-\frac{\partial \varphi_e}{\partial z}-\frac{\partial a_z}{\partial t}+u_r b_\theta-u_\theta b_r\right) \tag{14}$$

＜誘導方程式（B 表示）＞

$$\frac{\partial b_r}{\partial t}+u_r\frac{\partial b_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_r}{\partial \theta}+u_z\frac{\partial b_r}{\partial z}=b_r\frac{\partial u_r}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_r}{\partial \theta}+b_z\frac{\partial u_r}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_r}{\partial r}\right)-\frac{b_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 b_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial b_\theta}{\partial \theta}+\frac{\partial^2 b_r}{\partial z^2}\right) \tag{15}$$

$$\frac{\partial b_\theta}{\partial t}+u_r\frac{\partial b_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_\theta}{\partial \theta}+u_z\frac{\partial b_\theta}{\partial z}+\frac{u_\theta b_r}{r}=b_r\frac{\partial u_\theta}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+b_z\frac{\partial u_\theta}{\partial z}+\frac{b_\theta u_r}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_\theta}{\partial r}\right)-\frac{b_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 b_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial b_r}{\partial \theta}+\frac{\partial^2 b_\theta}{\partial z^2}\right) \tag{16}$$

$$\frac{\partial b_z}{\partial t}+u_r\frac{\partial b_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial b_z}{\partial \theta}+u_z\frac{\partial b_z}{\partial z}=b_r\frac{\partial u_z}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_z}{\partial \theta}+b_z\frac{\partial u_z}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 b_z}{\partial \theta^2}+\frac{\partial^2 b_z}{\partial z^2}\right) \tag{17}$$

＜誘導方程式（A 表示；クーロンゲージ）＞

$$\frac{\partial a_r}{\partial t}+\frac{u_\theta}{r}\frac{\partial a_r}{\partial \theta}+u_z\frac{\partial a_r}{\partial z}=-\frac{\partial \varphi_e}{\partial r}+u_\theta\frac{\partial a_\theta}{\partial r}+u_z\frac{\partial a_z}{\partial r}+\frac{u_\theta a_\theta}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_r}{\partial r}\right)-\frac{a_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 a_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial a_\theta}{\partial \theta}+\frac{\partial^2 a_r}{\partial z^2}\right) \tag{18}$$

$$\frac{\partial a_\theta}{\partial t}+u_r\frac{\partial a_\theta}{\partial r}+u_z\frac{\partial a_\theta}{\partial z}=-\frac{1}{r}\frac{\partial \varphi_e}{\partial \theta}+\frac{u_r}{r}\frac{\partial a_r}{\partial \theta}+\frac{u_z}{r}\frac{\partial a_z}{\partial \theta}-\frac{u_r a_\theta}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_\theta}{\partial r}\right)-\frac{a_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 a_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial a_r}{\partial \theta}+\frac{\partial^2 a_\theta}{\partial z^2}\right) \tag{19}$$

$$\frac{\partial a_z}{\partial t}+u_r\frac{\partial a_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial a_z}{\partial \theta}=-\frac{\partial \varphi_e}{\partial z}+u_r\frac{\partial a_r}{\partial z}+u_\theta\frac{\partial a_\theta}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 a_z}{\partial \theta^2}+\frac{\partial^2 a_z}{\partial z^2}\right) \tag{20}$$

＜磁荷不在の法則あるいはクーロンゲージ＞

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rb_r\right)+\frac{1}{r}\frac{\partial b_\theta}{\partial \theta}+\frac{\partial b_z}{\partial z}=0 \tag{21}$$

回転座標系

慣性座標系からの違いを色で強調して示す.

＜質量保存式＞

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial \hat{u}_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0 \tag{22}$$

＜運動方程式＞

$$\rho\left(\frac{\partial u_r}{\partial \hat{t}}+u_r\frac{\partial u_r}{\partial r}+\frac{\hat{u}_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{\hat{u}_\theta{}^2}{r}\right)\underbrace{-\rho r\Omega^2-2\rho\hat{u}_\theta\Omega}_{\text{centrifugal and Coriolis forces}}$$
$$=-\frac{\partial p}{\partial r}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial \hat{u}_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right) \tag{23}$$
$$+\frac{1}{\mu_m}\left(\frac{b_\theta}{r}\frac{\partial b_r}{\partial \theta}+b_z\frac{\partial b_r}{\partial z}-\frac{b_\theta b_\theta}{r}-b_\theta\frac{\partial b_\theta}{\partial r}-b_z\frac{\partial b_z}{\partial r}\right)+\rho g_r$$

$$\rho\left(\frac{\partial \hat{u}_\theta}{\partial \hat{t}}+u_r\frac{\partial \hat{u}_\theta}{\partial r}+\frac{\hat{u}_\theta}{r}\frac{\partial \hat{u}_\theta}{\partial \theta}+u_z\frac{\partial \hat{u}_\theta}{\partial z}+\frac{\hat{u}_\theta u_r}{r}\right)\underbrace{+2\rho\Omega u_r}_{\text{Coriolis force}}$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{u}_\theta}{\partial r}\right)-\frac{\hat{u}_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 \hat{u}_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 \hat{u}_\theta}{\partial z^2}\right) \tag{24}$$
$$+\frac{1}{\mu_m}\left(b_r\frac{\partial b_\theta}{\partial r}+b_z\frac{\partial b_\theta}{\partial z}+\frac{b_\theta b_r}{r}-\frac{b_r}{r}\frac{\partial b_r}{\partial \theta}-\frac{b_z}{r}\frac{\partial b_z}{\partial \theta}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial u_z}{\partial \hat{t}}+u_r\frac{\partial u_z}{\partial r}+\frac{\hat{u}_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right) \tag{25}$$
$$+\frac{1}{\mu_m}\left(b_r\frac{\partial b_z}{\partial r}+\frac{b_\theta}{r}\frac{\partial b_z}{\partial \theta}-b_r\frac{\partial b_r}{\partial z}-b_\theta\frac{\partial b_\theta}{\partial z}\right)+\rho g_z$$

＜エネルギー方程式＞

$$\frac{\partial T}{\partial \hat{t}}+u_r\frac{\partial T}{\partial r}+\frac{\hat{u}_\theta}{r}\frac{\partial T}{\partial \theta}+u_z\frac{\partial T}{\partial z}=\alpha\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right\} \tag{26}$$

＜電荷保存則＞

式(11)に同じ

＜Ohm の法則＞

$$j_r=\sigma\left(-\frac{\partial \hat{\varphi}_e}{\partial r}-\frac{\partial a_r}{\partial \hat{t}}+\hat{u}_\theta b_z-u_z b_\theta\right) \tag{27}$$

$$j_\theta=\sigma\left(-\frac{1}{r}\frac{\partial \hat{\varphi}_e}{\partial \theta}-\frac{\partial a_\theta}{\partial \hat{t}}+u_z b_r-u_r b_z\right) \tag{28}$$

$$j_z=\sigma\left(-\frac{\partial \hat{\varphi}_e}{\partial z}-\frac{\partial a_z}{\partial \hat{t}}+u_r b_\theta-\hat{u}_\theta b_r\right) \tag{29}$$

式(5)に示したように，回転系からみた電位にはベクトルポテンシャルの θ 成分を考慮することで，Ohm の法則は座標系に依らず不変となる．本稿では，以下の誘導方程式には磁場を用いる場合を紹介しているが，この方法は，Ohm の法則の Curl を取って，ベクトルポテンシャルと電位が現れないようにしたものである．一方で，Ohm の法則の Curl を取らずに，ベクトルポテンシャルと電位を同時に求めていく方法も提案できる．その場合は，得られた電位の値は，慣性系と回転系では異なることに注意を要する．

＜誘導方程式（B 表示）＞

$$\frac{\partial b_r}{\partial \hat{t}}+u_r\frac{\partial b_r}{\partial r}+\frac{\hat{u}_\theta}{r}\frac{\partial b_r}{\partial \theta}+u_z\frac{\partial b_r}{\partial z}=b_r\frac{\partial u_r}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_r}{\partial \theta}+b_z\frac{\partial u_r}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_r}{\partial r}\right)-\frac{b_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 b_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial b_\theta}{\partial \theta}+\frac{\partial^2 b_r}{\partial z^2}\right) \tag{30}$$

$$\frac{\partial b_\theta}{\partial \hat{t}}+u_r\frac{\partial b_\theta}{\partial r}+\frac{\hat{u}_\theta}{r}\frac{\partial b_\theta}{\partial \theta}+u_z\frac{\partial b_\theta}{\partial z}+\frac{\hat{u}_\theta b_r}{r}=b_r\frac{\partial \hat{u}_\theta}{\partial r}+\frac{b_\theta}{r}\frac{\partial \hat{u}_\theta}{\partial \theta}+b_z\frac{\partial \hat{u}_\theta}{\partial z}+\frac{b_\theta u_r}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_\theta}{\partial r}\right)-\frac{b_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 b_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial b_r}{\partial \theta}+\frac{\partial^2 b_\theta}{\partial z^2}\right) \tag{31}$$

$$\frac{\partial b_z}{\partial \hat{t}}+u_r\frac{\partial b_z}{\partial r}+\frac{\hat{u}_\theta}{r}\frac{\partial b_z}{\partial \theta}+u_z\frac{\partial b_z}{\partial z}=b_r\frac{\partial u_z}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_z}{\partial \theta}+b_z\frac{\partial u_z}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial b_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 b_z}{\partial \theta^2}+\frac{\partial^2 b_z}{\partial z^2}\right) \tag{32}$$

＜誘導方程式（A 表示；クーロンゲージ）＞

$$\frac{\partial a_r}{\partial \hat{t}}+\frac{\hat{u}_\theta}{r}\frac{\partial a_r}{\partial \theta}+u_z\frac{\partial a_r}{\partial z}=-\frac{\partial \hat{\varphi}_e}{\partial r}+\hat{u}_\theta\frac{\partial a_\theta}{\partial r}+u_z\frac{\partial a_z}{\partial r}+\frac{\hat{u}_\theta a_\theta}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_r}{\partial r}\right)-\frac{a_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 a_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial a_\theta}{\partial \theta}+\frac{\partial^2 a_r}{\partial z^2}\right) \tag{33}$$

$$\frac{\partial a_\theta}{\partial \hat{t}}+u_r\frac{\partial a_\theta}{\partial r}+u_z\frac{\partial a_\theta}{\partial z}=-\frac{1}{r}\frac{\partial \hat{\varphi}_e}{\partial \theta}+\frac{u_r}{r}\frac{\partial a_r}{\partial \theta}+\frac{u_z}{r}\frac{\partial a_z}{\partial \theta}-\frac{u_r a_\theta}{r}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_\theta}{\partial r}\right)-\frac{a_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 a_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial a_r}{\partial \theta}+\frac{\partial^2 a_\theta}{\partial z^2}\right) \tag{34}$$

$$\frac{\partial a_z}{\partial \hat{t}}+u_r\frac{\partial a_z}{\partial r}+\frac{\hat{u}_\theta}{r}\frac{\partial a_z}{\partial \theta}=-\frac{\partial \hat{\varphi}_e}{\partial z}+u_r\frac{\partial a_r}{\partial z}+\hat{u}_\theta\frac{\partial a_\theta}{\partial z}$$
$$+\nu_m\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 a_z}{\partial \theta^2}+\frac{\partial^2 a_z}{\partial z^2}\right) \tag{35}$$

＜磁荷不在の法則＞

式(21)に同じ.

以上をまとめると，時間項のある方程式に対しても，式(4)の回転座標系における時間を導入すれば，運動方程式以外の式は不変である．運動方程式にはコリオリ力と遠心力の項が追加で現れる．実際の回転座標系における熱対流の数値計算には，式(19), (20)の重力項や遠心力項に対して，Boussinesq 近似等を用いれば良い.

付録　式(4)について

<u>１．ベクトル値関数の時間変化</u>

ベクトル$\vec{a}$の時間変化の場合は

$$\frac{\partial \vec{a}}{\partial \hat{t}}=\frac{\partial \vec{a}}{\partial t}+\left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}\vec{a}-\vec{\Omega}\times\vec{a}$$

と書ける．これが，

$$\frac{\partial a_S}{\partial \hat{t}}=\frac{\partial a_S}{\partial t}+\Omega\frac{\partial a_S}{\partial \theta},\qquad \left(a_S=a_r,a_\theta,a_z\right)$$

に一致することについて以下に示す．まず，右辺第 3 項は

$$\vec{\Omega}\times\vec{a}=\begin{vmatrix}\vec{e}_r & \vec{e}_\theta & \vec{e}_z\\ 0 & 0 & \Omega\\ a_r & a_\theta & a_z\end{vmatrix}=\left(-a_\theta\Omega\right)\vec{e}_r+\left(a_r\Omega\right)\vec{e}_\theta+\left(0\right)\vec{e}_z$$

と求まる．次に，右辺第２項の$\left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}\vec{a}$ に対しては，以下の公式を利用する．

$$\vec{b}\cdot\vec{\nabla}\vec{u}=\left(b_r\frac{\partial u_r}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{b_\theta u_\theta}{r}+b_z\frac{\partial u_r}{\partial z}\right)\vec{e}_r+\left(b_r\frac{\partial u_\theta}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{b_\theta u_r}{r}+b_z\frac{\partial u_\theta}{\partial z}\right)\vec{e}_\theta$$
$$+\left(b_r\frac{\partial u_z}{\partial r}+\frac{b_\theta}{r}\frac{\partial u_z}{\partial\theta}+b_z\frac{\partial u_z}{\partial z}\right)\vec{e}_z$$

ここで，$\vec{b}=\vec{\Omega}\times\vec{x}$，$\vec{u}=\vec{a}$ と置く．ただし，$\vec{\Omega}\times\vec{x}=\left(\Omega r\right)\vec{e}_\theta$　に留意する．

$$\left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}\vec{a}=\left(\frac{\Omega r}{r}\frac{\partial a_r}{\partial\theta}-\frac{\Omega r a_\theta}{r}\right)\vec{e}_r+\left(\frac{\Omega r}{r}\frac{\partial a_\theta}{\partial\theta}+\frac{\Omega r a_r}{r}\right)\vec{e}_\theta+\left(\frac{\Omega r}{r}\frac{\partial a_z}{\partial\theta}\right)\vec{e}_z$$
$$\therefore\ \left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}\vec{a}=\left(\Omega\frac{\partial a_r}{\partial\theta}-\Omega a_\theta\right)\vec{e}_r+\left(\Omega\frac{\partial a_\theta}{\partial\theta}+\Omega a_r\right)\vec{e}_\theta+\left(\Omega\frac{\partial a_z}{\partial\theta}\right)\vec{e}_z$$

２つの項を同時に考慮すると，

$$\left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}\vec{a}-\vec{\Omega}\times\vec{a}=\left(\Omega\frac{\partial a_r}{\partial\theta}\right)\vec{e}_r+\left(\Omega\frac{\partial a_\theta}{\partial\theta}\right)\vec{e}_\theta+\left(\Omega\frac{\partial a_z}{\partial\theta}\right)\vec{e}_z$$

したがって，

$$\frac{\partial\vec{a}}{\partial\widehat{t}}=\frac{\partial\vec{a}}{\partial t}+\left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}\vec{a}-\vec{\Omega}\times\vec{a}$$
$$=\underbrace{\left(\frac{\partial a_r}{\partial t}+\Omega\frac{\partial a_r}{\partial\theta}\right)}_{\frac{\partial a_r}{\partial\widehat{t}}}\vec{e}_r+\underbrace{\left(\frac{\partial a_\theta}{\partial t}+\Omega\frac{\partial a_\theta}{\partial\theta}\right)}_{\frac{\partial a_\theta}{\partial\widehat{t}}}\vec{e}_\theta+\underbrace{\left(\frac{\partial a_z}{\partial t}+\Omega\frac{\partial a_z}{\partial\theta}\right)}_{\frac{\partial a_z}{\partial\widehat{t}}}\vec{e}_z$$

となる．

２．スカラ値関数の時間変化

スカラの時間変化では$\vec{\Omega}\times$の項は省かれ

$$\frac{\partial\varphi}{\partial\widehat{t}}=\frac{\partial\varphi}{\partial t}+\left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}\varphi$$

と書ける．これが，

$$\frac{\partial\varphi}{\partial\widehat{t}}=\frac{\partial\varphi}{\partial t}+\Omega\frac{\partial\varphi}{\partial\theta}$$

に一致するかどうか調べる．

$$\left(\vec{\Omega}\times\vec{x}\right)\cdot\vec{\nabla}=\left(\vec{e}_\theta\Omega r\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)=\Omega\frac{\partial}{\partial\theta}$$

したがって，

$$\frac{\partial \varphi}{\partial \widehat{t}} = \frac{\partial \varphi}{\partial t} + \left(\vec{\Omega} \times \vec{x}\right) \cdot \vec{\nabla} \varphi = \frac{\partial \varphi}{\partial t} + \Omega \frac{\partial \varphi}{\partial \theta}$$

となる.

結局，この問題では，ベクトルあるいはスカラのどちらの関数が作用する場合でも，この回転系での時間変化は

$$\frac{\partial}{\partial \widehat{t}} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}$$

と表される.